

# Bifurcation of critical points along gap-continuous families of subspaces

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*Dedicated to our friend Alessandro Portaluri  
Get well very soon!*

## Abstract

We consider the restriction of twice differentiable functionals on a Hilbert space to families of subspaces that vary continuously with respect to the gap metric. We study bifurcation of branches of critical points along these families, and apply our results to semilinear systems of ordinary differential equations.

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## 1 Introduction

Let  $H$  be a real separable Hilbert space and  $\mathcal{J} : H \rightarrow \mathbb{R}$  a  $C^2$ -functional. We denote the derivative of  $\mathcal{J}$  at  $u \in H$  by  $d_u \mathcal{J} \in \mathcal{L}(H, \mathbb{R})$  and, henceforth, assume that  $d_0 \mathcal{J} = 0$ , i.e.  $0 \in H$  is a critical point of  $\mathcal{J}$ . Usually, critical points of a functional  $\mathcal{J}$  on a suitable  $H$  are studied since they correspond to solutions of a related differential equation. Accordingly, critical points of a restriction  $\mathcal{J}|_{H'} : H' \rightarrow \mathbb{R}$  may correspond to solutions of such a differential equation but under additional constraints.

In [1] Abbondandolo and Majer study the *Grassmannian* of a Hilbert space  $H$ , i.e. the set of all closed subspaces of  $H$ . On this set there is a canonical metric which is induced by orthogonal projections, and consequently we can define paths  $\{H_t\}_{t \in [a, b]}$  in it. Using that for each  $t \in [a, b]$  the element  $0 \in H_t$  is a critical point of the restriction  $\mathcal{J}|_{H_t} : H_t \rightarrow \mathbb{R}$ , the goal of this paper is to study bifurcations from this branch of critical points (for the formal definition, see Definition 3.1).

Here, we provide existence results for bifurcations in terms of the second derivative of  $\mathcal{J}$  at the critical point  $0$ , which are based on [8] and [15]. To this aim, we introduce a family of functionals  $f_t : H \rightarrow \mathbb{R}$ ,  $t \in [a, b]$ , such that each  $f_t$  involves the orthogonal projection onto the space  $H_t$ , and is such that its critical points are the critical points of the restriction  $\mathcal{J}|_{H_t}$ . Consequently,  $0 \in H$  is a critical point of any  $f_t : H \rightarrow \mathbb{R}$ ,  $t \in [a, b]$ , and, by considering the second derivative  $d_0^2 f_t$  of  $f_t$  at  $0$  we can define a path  $\{L_t\}_{t \in [a, b]}$  of bounded selfadjoint operators by the Riesz Representation Theorem. The assumptions of our theorems ensure that each  $L_t$  is actually a Fredholm operator, and we prove that bifurcation of critical points of  $f$  along  $\{H_t\}_{t \in [a, b]}$  arise if the *spectral flow* of  $L : t \mapsto L_t$  does not vanish. Let us recall that the spectral flow is an integer

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valued homotopy invariant for paths of selfadjoint Fredholm operators that was introduced by Atiyah, Patodi and Singer in [3], and whose relevance in bifurcation theory was discovered in [8]. For example, if all operators  $L_t$  have a finite Morse index  $\mu_{Morse}(L_t)$ , then the spectral flow of  $L$  is just the difference of the Morse indices at the endpoints, i.e.  $\mu_{Morse}(L_a) - \mu_{Morse}(L_b)$ . Then a non-vanishing spectral flow of  $L$  corresponds to a jump in the Morse indices of  $L$ , which implies bifurcation of critical points of  $f$  by a well known theorem in bifurcation theory (cf. [14, §8.9], [11, §II.7.1]). However, if  $\mu_{Morse}(L_t) = +\infty$  for some  $t \in [a, b]$ , then the spectral flow may depend on the whole path  $L$  and not only on its endpoints, which makes the theory more complicated.

The paper is structured as follows. In Section 2, we introduce some preliminaries that we need in order to state our theorems. First, we recall some facts about the Grassmannian of a Hilbert space  $H$ , essentially following Abbondandolo and Majer's paper [1]. However, we also state and prove a folklore result which shows that the kernels of families of surjective bounded operators on  $H$  yield paths in the Grassmannian and which we use in the final section for creating examples. Second, we briefly recall the definition of the spectral flow from [8]. In Section 3, we introduce the path  $L$  and state our main theorems and a corollary, which we prove in Section 4. Finally, in Section 5 we apply our theory to a Dirichlet problem for indefinite semilinear ordinary differential operators.

## 2 Grassmannians and spectral flows

Here and in the following, let  $H$  be a real separable Hilbert space of infinite dimension, and we denote by  $\mathcal{L}(H)$  the Banach space of all linear bounded operators on  $H$  and by  $I_H \in \mathcal{L}(H)$  the identity operator.

Let us recall that a *Fredholm operator*  $T$  on a Hilbert space  $H$  is an operator  $T \in \mathcal{L}(H)$  such that both its kernel and its cokernel are of finite dimension. We denote the open subset of all Fredholm operators in  $\mathcal{L}(H)$  by  $\Phi(H)$ . Furthermore, a *path*  $\{K_t\}_{t \in [a, b]}$  in a metric space  $\mathcal{G}$  is a continuous map  $K : t \in [a, b] \mapsto K_t \in \mathcal{G}$ .

### 2.1 The Grassmannian of a Hilbert space

In this section, we recall briefly the definition and some properties of the *Grassmannian*  $\mathcal{G}(H)$  of  $H$ , i.e. the set of all closed linear subspaces of  $H$  (for all the details see the comprehensive exposition [1]).

For every  $U \in \mathcal{G}(H)$ , there exists a unique orthogonal projection  $P_U : H \rightarrow H$  onto  $U$  and the distance

$$d(U, V) := \|P_U - P_V\|, \quad U, V \in \mathcal{G}(H),$$

makes  $\mathcal{G}(H)$  into a complete metric space (cf. also [10]). Moreover, one can show that  $\mathcal{G}(H)$  is an analytical Banach manifold, and the map

$$V \in \mathcal{G}(H) \mapsto P_V \in \mathcal{L}(H)$$

embeds  $\mathcal{G}(H)$  analytically into  $\mathcal{L}(H)$  (cf. [1, Proposition 1.1]).

**Lemma 2.1.** *The connected components of  $\mathcal{G}(H)$  are the sets*

$$\mathcal{G}_{nk}(H) = \{V \in \mathcal{G}(H) : \dim V = n, \text{codim } V = k\},$$

with  $n, k \in \mathbb{N} \cup \{+\infty\}$  such that  $k + n = +\infty$ .

*Proof.* Firstly, we note that, if  $\|P_U - P_V\| < 1$  for  $U, V \in \mathcal{G}(H)$ , then  $\dim U = \dim V$  and  $\dim U^\perp = \dim V^\perp$  (cf. [10, I.4.6]). Consequently, if  $U$  and  $V$  belong to the same component of  $\mathcal{G}(H)$ , then they must have both the same dimension and the same codimension.

Now, let us assume that  $U, V \in \mathcal{G}_{nk}(H)$  for some  $k, n$  such that  $k+n = +\infty$ . Since  $H$  is separable, it is easy to construct an orthogonal operator  $O : H \rightarrow H$  such that  $O(U) = V$ . Denoting by  $\mathcal{O}(H)$  the subspace of  $\mathcal{L}(H)$  consisting of all the orthogonal operators, some tools in Functional Calculus allow one to show that  $\mathcal{O}(H)$  is connected<sup>1</sup>. Hence, there is a path  $\{O_t\}_{t \in [0,1]} \subset \mathcal{O}(H)$  joining the identity operator  $I_H$  to  $O$ .

Finally, since  $P_{O_t(U)} = O_t^{-1} P_U O_t$  for each  $t \in [0, 1]$ , we have that  $\{O_t(U)\}_{t \in [0,1]}$  is a path in  $\mathcal{G}(H)$  that joins  $U$  to  $V$ .  $\square$

Let us point out that a computation of all homotopy groups  $\pi_i(\mathcal{G}_{nk}(H))$  can be found in [1, Section 2].

The following lemma is essentially well known (e.g., cf. [6, Appendix A]), but we are not aware of a proof in the literature, and so we include it here for the sake of completeness. However, the reader may compare it with a related assertion on Banach bundles, which can be found e.g. in [23] (cf. also [21]), and on which our argument is based.

**Lemma 2.2.** *Let  $A : [a, b] \times H \rightarrow X$  be a continuous family of surjective maps, where  $X$  is some Banach space. Then*

$$\{\ker A_t\}_{t \in [a,b]} := \{u \in H : A_t u = 0\}_{t \in [a,b]}$$

*is a path in  $\mathcal{G}_{nk}(H)$ , where  $k = \dim X$ .*

*Proof.* First, let us fix some  $t_0 \in [a, b]$ . Since  $A_{t_0}$  is surjective, there exists  $M_0 \in \mathcal{L}(X, H)$  such that  $A_{t_0} M_0 = I_X$ . From the fact that the invertible elements in  $\mathcal{L}(X)$  are open, we see that  $A_t M_0$  is invertible for all  $t$  in a neighbourhood  $U_0$  of  $t_0$ .

Now, if we set  $M_{0,t} := M_0 (A_t M_0)^{-1}$  for  $t \in U_0$ , then  $A_t M_{0,t} = I_X$ .

Note that if  $M_1, M_2 \in \mathcal{L}(X, H)$  are such that  $A_t M_i = I_X$ , then  $A_t(\alpha M_1 + (1 - \alpha) M_2) = I_X$  for all  $0 \leq \alpha \leq 1$ . Consequently, by using a partition of unity, we may conclude that there exists a family  $M : [a, b] \rightarrow \mathcal{L}(X, H)$  such that  $A_t M_t = I_X$  for all  $t \in [a, b]$ .

Defining  $R_t := M_t A_t \in \mathcal{L}(H)$ , we note that  $R_t$  is a projection since

$$R_t^2 = M_t A_t M_t A_t = M_t A_t = R_t.$$

Moreover, since  $M_t$  is clearly injective, we infer that

$$\ker R_t = \ker(M_t A_t) = \ker(A_t)$$

so that  $Q_t := I_H - R_t$  is a continuous family of projections such that  $\text{im } Q_t = \ker A_t$ . Thus, taking

$$P_t = Q_t Q_t^* (Q_t Q_t^* + (I_H - Q_t^*)(I_H - Q_t))^{-1},$$

by [5, Lemma 12.8 a)] it follows that  $\{P_t\}_{t \in [a,b]}$  is a continuous family of orthogonal projections such that  $\text{im } P_t = \ker A_t$ ; hence,  $\{\ker A_t\}_{t \in [a,b]}$  is a continuous family of subspaces in  $\mathcal{G}(H)$ .

Finally,  $\ker A_t \in \mathcal{G}_{nk}(H)$  with  $k = \dim X$ , is an immediate consequence of the rank-nullity theorem in Linear Algebra.  $\square$

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<sup>1</sup>Indeed, even more is true: in [12] Kuiper proves that  $\mathcal{O}(H)$  is contractible.

## 2.2 The spectral flow

We denote by  $\Phi_S(H) \subset \Phi(H)$  the subspace of all selfadjoint Fredholm operators, which is well known to consist of three connected components (cf. [4]). Two of them are given by

$$\begin{aligned}\Phi_S^+(H) &= \{L \in \Phi_S(H) : \sigma_{ess}(L) \subset (0, +\infty)\}, \\ \Phi_S^-(H) &= \{L \in \Phi_S(H) : \sigma_{ess}(L) \subset (-\infty, 0)\},\end{aligned}$$

where  $\sigma_{ess}(L) = \{\lambda \in \mathbb{R} : L - \lambda I_H \notin \Phi_S(H)\}$  is the *essential spectrum* of an operator  $L \in \Phi_S(H)$ . Their elements are called *essentially positive* or *essentially negative*, respectively, and it is readily seen that both of these spaces are contractible. Elements of the remaining component  $\Phi_S^i(H) = \Phi_S(H) \setminus (\Phi_S^+(H) \cup \Phi_S^-(H))$  are called *strongly indefinite*, and, in contrast to  $\Phi_S^+(H)$  and  $\Phi_S^-(H)$ , this space has a non-trivial topology. Indeed,  $\Phi_S^i(H)$  has the same homotopy groups as the stable orthogonal group (cf. [20]) and the spectral flow provides an explicit isomorphism between its fundamental group and the integers. There are several different, but equivalent, constructions of the spectral flow in the literature. Here, we follow the approach developed by Fitzpatrick, Pejsachowicz and Recht in [8], and we refer to the introduction of [15] for further references on the subject.

We call two selfadjoint invertible operators in  $\mathcal{L}(H)$  *Calkin equivalent* if  $S - T$  is compact. It is well known that in this case the *relative Morse index*

$$\mu_{rel}(S, T) = \dim(E^-(S) \cap E^+(T)) - \dim(E^+(S) \cap E^-(T))$$

is well defined and finite, where  $E^-(\cdot)$  and  $E^+(\cdot)$  denote the negative and positive subspaces of a selfadjoint operator for which 0 is an isolated point of the spectrum.

From the second resolvent identity it follows that for Calkin equivalent operators  $S, T$ , also the difference of the associated resolvent operators

$$(\lambda - T)^{-1} - (\lambda - S)^{-1} = (\lambda - T)^{-1}(S - T)(\lambda - S)^{-1}, \quad \lambda \notin \sigma(T) \cup \sigma(S),$$

is compact whenever it is defined, where  $\sigma(T)$  and  $\sigma(S)$  denote the spectrum of  $T$ , respectively  $S$ . Finally, since the compact operators are closed in  $\mathcal{L}(H)$ , it may be concluded that also the difference of the spectral projections

$$P_{[a,b]}(T) - P_{[a,b]}(S) = \operatorname{Re} \left( \frac{1}{2\pi i} \int_{\Gamma} [(\lambda - T^{\mathbb{C}})^{-1} - (\lambda - S^{\mathbb{C}})^{-1}] d\lambda \right)$$

is compact, where  $a, b$  do not belong to  $\sigma(S) \cup \sigma(T)$  and  $\Gamma$  is the circle around  $\frac{a+b}{2}$  in  $\mathbb{C}$  intersecting the real axis at  $a$  and  $b$ . Here,  $S^{\mathbb{C}}$  and  $T^{\mathbb{C}}$  denote the complexification of operators and  $\operatorname{Re}$  the real part of an operator on a complexified Hilbert space (cf. [22, Subsection 2.1] for more details). The group  $GL(H)$  of all invertible operators on  $H$  acts on  $\Phi_S(H)$  by mapping  $M \in GL(H)$  to  $M^*LM$ , with  $L \in \Phi_S(H)$ , which is called the *cogredient action*. One of the main theorems in [8] states that for any path  $L : [a, b] \rightarrow \Phi_S(H)$  there exist a path  $M : [a, b] \rightarrow GL(H)$  and an invertible operator  $J \in \Phi_S(H)$ , such that  $M_t^* L_t M_t = J + K_t$  with  $K_t$  compact for each  $t \in [a, b]$ .

**Definition 2.3.** Let  $L : [a, b] \rightarrow \Phi_S(H)$  be a path such that  $L_a$  and  $L_b$  are invertible. The *spectral flow* of  $L$  is the integer

$$\operatorname{sf}(L, [a, b]) = \mu_{rel}(J + K_a, J + K_b),$$

where  $\{J + K_t\}_{t \in [a, b]}$  is any path of compact perturbations of an invertible operator  $J \in \Phi_S(H)$  which is cogredient with  $\{L_t\}_{t \in [a, b]}$ .

From well known properties of the relative Morse index it follows that the spectral flow does not depend on the choices of  $J$  and  $\{K_t\}_{t \in [a,b]}$ , and has the following main properties:

- (i) if  $L_t$  is invertible for all  $t \in [a, b]$ , then  $\text{sf}(L, [a, b]) = 0$ ;
- (ii) if  $H_1$  and  $H_2$  are separable Hilbert spaces and the paths  $L_1 : [a, b] \rightarrow \Phi_S(H_1)$  and  $L_2 : [a, b] \rightarrow \Phi_S(H_2)$  have invertible ends, then

$$\text{sf}(L_1 \oplus L_2, [a, b]) = \text{sf}(L_1, [a, b]) + \text{sf}(L_2, [a, b]);$$

- (iii) let  $h : [0, 1] \times [a, b] \rightarrow \Phi_S(H)$  be a homotopy such that  $h(s, a)$  and  $h(s, b)$  are invertible for all  $s \in [0, 1]$ . Then,

$$\text{sf}(h(0, \cdot), [a, b]) = \text{sf}(h(1, \cdot), [a, b]);$$

- (iv) if  $L_t \in \Phi_S^+(H)$ ,  $t \in [a, b]$ , and  $L_a, L_b$  are invertible, then the spectral flow of  $L$  is the difference of the Morse indices at its endpoints:

$$\text{sf}(L, [a, b]) = \mu_{\text{Morse}}(L_a) - \mu_{\text{Morse}}(L_b),$$

where

$$\mu_{\text{Morse}}(L_t) = \sup \dim \{V \subset H : \langle L_t u, u \rangle_H < 0 \text{ for all } u \in V \setminus \{0\}\}.$$

Finally, let us remark that the spectral flow is even uniquely characterised by the properties i)–iv) above (cf. [7]). A further uniqueness theorem for the spectral flow, which is based on the different but equivalent construction [16], can be found in [13, Subsection 5.2].

### 3 Bifurcation along gap continuous paths of subspaces

Throughout this section, let  $H$  be a real Hilbert space and  $\mathcal{J} : H \rightarrow \mathbb{R}$  a  $C^2$  functional having 0 as a critical point. We denote by  $d_u \mathcal{J} \in \mathcal{L}(H, \mathbb{R})$  the derivative of  $\mathcal{J}$  at  $u \in H$ . Moreover, let  $T$  be the Riez representation of the Hessian  $d_0^2 \mathcal{J} : H \times H \rightarrow \mathbb{R}$  of  $\mathcal{J}$  at 0, i.e., the unique selfadjoint operator  $T \in \mathcal{L}(H)$  which satisfies

$$d_0^2 \mathcal{J}[u, v] = \langle Tu, v \rangle_H, \quad u, v \in H. \quad (3.1)$$

Let  $\{H_t\}_{t \in [a,b]} \subset \mathcal{G}(H)$  be a gap continuous path of closed subspaces of  $H$  for some real numbers  $a < b$ , and let us point out that  $0 \in H$  is contained in any  $H_t$ ,  $t \in [a, b]$ . In what follows we denote by  $\mathcal{J}|_{H_t} : H_t \rightarrow \mathbb{R}$  the restriction of the functional  $\mathcal{J}$  to the closed subspace  $H_t \subset H$ . Note that  $0 \in H$  is a critical point of all  $\mathcal{J}|_{H_t}$ ,  $t \in [a, b]$ , which is a direct consequence of the uniqueness of the derivative.

**Definition 3.1.** We say that  $t^* \in [a, b]$  is a *bifurcation point of  $\mathcal{J}$  along  $\{H_t\}_{t \in [a,b]}$*  if there exist two sequences  $(t_n)_n \subset [a, b]$  and  $(u_n)_n \subset H$  such that

- i)  $t_n \rightarrow t^*$  in  $[a, b]$  and  $u_n \rightarrow 0$  in  $H$  as  $n \rightarrow +\infty$ ;
- ii)  $u_n \in H_{t_n}$  and  $u_n \neq 0$  for all  $n \in \mathbb{N}$ ;
- iii)  $u_n$  is a critical point of  $\mathcal{J}|_{H_{t_n}}$  for all  $n \in \mathbb{N}$ .

Since  $\{H_t\}_{t \in [a,b]}$  is a continuous path of subspaces, there exists a family  $P_t$ ,  $t \in [a,b]$ , of orthogonal projections such that  $\text{im } P_t = H_t$ . We define

$$L_t = P_t T P_t + P_t^\perp \quad \text{for each } t \in [a,b], \quad (3.2)$$

which is a continuous path of selfadjoint operators in  $\mathcal{L}(H)$ , and call  $\{H_t\}_{t \in [a,b]}$  *admissible* if both the operators

$$P_a T P_a : H_a \rightarrow H_a \quad \text{and} \quad P_b T P_b : H_b \rightarrow H_b$$

are invertible. Since  $H_t$  and  $H_t^\perp$  reduce  $L_t$ , and  $L_t|_{H_t^\perp} = I_{H_t^\perp}$  is invertible, we see at once that  $L_a$  and  $L_b$  are invertible if  $\{H_t\}_{t \in [a,b]}$  is admissible.

Now, let us state our main theorems and a corollary, which we are proving in the next section.

**Theorem 3.2.** *Let  $\{H_t\}_{t \in [a,b]}$  be an admissible path in  $\mathcal{G}_{nk}(H)$  such that either  $n \neq +\infty$  or  $k \neq +\infty$ , and let us assume that the operator  $T$  introduced in (3.1) is Fredholm. If  $\text{sf}(L, [a,b]) \neq 0$ , then there is a bifurcation point of  $\mathcal{J}$  along  $\{H_t\}_{t \in [a,b]}$ . Moreover, if  $n \neq +\infty$  and  $\{H_t\}_{t \in [a,b]}$  is analytic, then there are at least*

$$\left\lfloor \frac{|\text{sf}(L, [a,b])|}{n} \right\rfloor \quad (3.3)$$

*distinct bifurcation points (here,  $\lfloor \cdot \rfloor$  represents the integer part of a positive real number).*

The case in which the path  $\{H_t\}_{t \in [a,b]}$  is in the connected component  $\mathcal{G}_{nk}(H)$  of  $\mathcal{G}(H)$  with both  $n$  and  $k$  infinite, is not allowed in Theorem 3.2. In our second theorem we consider this setting but we have to pay a restriction on the form of the operator  $T$ .

**Theorem 3.3.** *We assume that  $T = I_H + K$  for some compact operator  $K$ , and  $\{H_t\}_{t \in [a,b]}$  is an admissible path in  $\mathcal{G}_{\infty,\infty}(H)$ . If  $\text{sf}(L, [a,b]) \neq 0$ , then there is a bifurcation point of  $\mathcal{J}$  along  $\{H_t\}_{t \in [a,b]}$ .*

Let us point out that  $L_t \in \Phi_S^+(H)$ ,  $t \in [a,b]$ , and so

$$\text{sf}(L, [a,b]) = \mu_{\text{Morse}}(L_a) - \mu_{\text{Morse}}(L_b) = \mu_{\text{Morse}}(T|_{H_a}) - \mu_{\text{Morse}}(T|_{H_b}),$$

in any of the following cases:

- if in Theorem 3.2 it is  $n \neq +\infty$ , since each  $L_t$  is positive on the subspace  $H_t^\perp$  which is of finite codimension;
- if in Theorem 3.2 it is  $T \in \Phi_S^+(H)$ , since  $\mu_{\text{Morse}}(L_t) \leq \mu_{\text{Morse}}(T)$  for all  $t \in [a,b]$ ;
- for all compact operator  $K$  in Theorem 3.3 by the same argument as in the previous item.

Finally, we will prove in the subsequent section a corollary of the proof of Theorem 3.2, which rephrases a well known fact from bifurcation theory in our setting. Let us point out that both Theorem 3.2 and Theorem 3.3 do not give any information about the location of the bifurcation point in the interval  $(a,b)$ .

**Corollary 3.4.** *We assume that either the assumptions of Theorem 3.2 or the ones of Theorem 3.3 hold. If  $t^*$  is a bifurcation point, then*

$$\text{im}(T|_{H_{t^*}}) \cap H_{t^*}^\perp \neq \{0\}.$$

## 4 Proofs of the main theorems

Our proofs are based on the main theorem of [15], which deals with the relation between the spectral flow and the bifurcation theory that was previously established in [8]. For completeness, we recall it.

Let  $f : [a, b] \times H \rightarrow \mathbb{R}$  be a continuous map such that each  $f_t := f(t, \cdot)$  is  $C^2$  and all its derivatives depend continuously on  $t \in [a, b]$ . Henceforth, we assume that  $0 \in H$  is a critical point of all  $f_t$ , and we call  $t^*$  a *bifurcation point of critical points of the functional  $f$*  if there exist two sequences  $(t_n)_n \subset [a, b]$  and  $(u_n)_n \subset H \setminus \{0\}$  such that  $t_n \rightarrow t^*$  in  $[a, b]$ ,  $u_n \rightarrow 0$  in  $H$  and  $u_n$  is a critical point of  $f_{t_n}$  for all  $n \in \mathbb{N}$ .

The second derivatives of  $f_t$ ,  $t \in [a, b]$ , define a path of selfadjoint operators and we denote the corresponding Riesz representations of  $d_0^2 f_t$  by  $L_t$ . The bifurcation theorem in [15] can be stated as follows:

**Theorem 4.1.** *If each  $L_t$ ,  $t \in [a, b]$ , is a Fredholm operator, both  $L_a$  and  $L_b$  are invertible and  $\text{sf}(L, [a, b]) \neq 0$ , then there is a bifurcation point of critical points of the functional  $f$  in  $(a, b)$ . Moreover, if there are only finitely many  $t \in (a, b)$  such that  $\ker L_t \neq 0$  and*

$$m := \sup_{t \in (a, b)} \dim \ker L_t < +\infty,$$

*then the number of bifurcation points is at least*

$$\left\lfloor \frac{|\text{sf}(L, [a, b])|}{m} \right\rfloor.$$

Now, in the setting of Section 3 we define a one-parameter family of functionals by

$$f_t : u \in H \mapsto f_t(u) = \mathcal{J}(P_t u) + \frac{1}{2} \|P_t^\perp u\|^2 \in \mathbb{R}.$$

**Lemma 4.2.** *The critical points of  $f_t$  are precisely the critical points of  $\mathcal{J}|_{H_t}$ ,  $t \in [a, b]$ .*

*Proof.* If  $u$  is a critical point of  $f_t$ , then

$$d_u f_t(v) = d_{P_t u} \mathcal{J}(P_t v) + \langle P_t^\perp u, P_t^\perp v \rangle = 0 \quad \text{for all } v \in H. \quad (4.1)$$

In particular, taking  $v = P_t^\perp u$ , it is

$$0 = d_{P_t u} \mathcal{J}(P_t P_t^\perp u) + \|P_t^\perp u\|^2 \quad \text{with } P_t P_t^\perp u = 0,$$

hence,  $P_t^\perp u = 0$  and so  $u \in H_t$ . Consequently, from (4.1) we obtain that

$$0 = d_{P_t u} \mathcal{J}(P_t v) = d_u \mathcal{J}(v) \quad \text{for all } v \in H_t;$$

whence,  $u$  is a critical point of the restriction of  $\mathcal{J}$  to  $H_t$ .

Conversely, if  $u$  is a critical point of the restriction of  $\mathcal{J}$  to  $H_t$ , then  $u \in H_t$  and

$$d_u f_t(v) = d_{P_t u} \mathcal{J}(P_t v) + \langle P_t^\perp u, P_t^\perp v \rangle = d_u \mathcal{J}(P_t v)$$

which vanishes for all  $v \in H$  since  $P_t v \in H_t$ . □

Consequently, from Definition 3.1 and Lemma 4.2 it follows that  $t^* \in [a, b]$  is a bifurcation point of  $\mathcal{J}$  along  $\{H_t\}_{t \in [a, b]}$  if and only if it is a bifurcation point for the family of functionals  $f_t$ .

By applying Theorem 4.1, for each  $t \in [a, b]$  we have to consider the Hessian of  $f_t$  at the critical point  $0 \in H$ , which is given by

$$d_0^2 f_t[u, v] = d_0^2 \mathcal{J}[P_t u, P_t v] + \langle P_t^\perp u, P_t^\perp v \rangle \quad \text{for all } u, v \in H,$$

and which has the Riesz representation

$$L_t = P_t^* T P_t + P_t^\perp.$$

Note that  $\{L_t\}_{t \in [a, b]}$  are exactly the operators introduced in (3.2).

Now, we deduce Theorems 3.2 and 3.3 from Theorem 4.1 but before we point out a direct consequence of the definition of Fredholm operators.

**Lemma 4.3.** *If  $H_1, H_2$  are Hilbert spaces and  $T_1 : H_1 \rightarrow H_1, T_2 : H_2 \rightarrow H_2$  are Fredholm operators, then*

$$T_1 \oplus T_2 : u_1 + u_2 \in H_1 \oplus H_2 \mapsto (T_1 \oplus T_2)(u_1 + u_2) = T_1 u_1 + T_2 u_2 \in H_1 \oplus H_2$$

*is a Fredholm operator of index  $\text{ind}(T_1 \oplus T_2) = \text{ind}(T_1) + \text{ind}(T_2)$ .*

In what follows, we will apply Lemma 4.3 to  $L_t|_{H_t} : H_t \rightarrow H_t$  and  $L_t|_{H_t^\perp} : H_t^\perp \rightarrow H_t^\perp$ .

*Proof of Theorem 3.2.* Firstly, assume that  $n \neq +\infty$ . Then, by Lemma 4.3 the operator  $L_t$  is Fredholm, since it is invertible on the subspace  $H_t^\perp$  and Fredholm on the finite dimensional space  $H_t$ . Furthermore, by assumption  $L_a$  and  $L_b$  are invertible, and so it is enough to apply Theorem 4.1 in order to conclude that  $f$  admits a bifurcation point.

Moreover, if  $\{H_t\}_{t \in [a, b]}$  is analytic, then  $P_t$  and so  $L_t$  depend analytically on  $t$ . Reasoning as in [15, Section 2] the set of all  $t$  such that  $\ker L_t \neq \{0\}$  is discrete. Moreover, it is readily seen that

$$\ker L_t = \text{im}(T|_{H_t}) \cap H_t^\perp, \quad (4.2)$$

for any  $t \in [a, b]$ , which implies that

$$\dim \ker L_t \leq \dim \text{im}(T|_{H_t}) \leq \dim H_t = n.$$

Hence, (3.3) follows from Theorem 4.1.

On the other hand, assume that  $k \neq +\infty$ . Since again  $L_a$  and  $L_b$  are invertible by assumption, in order to apply Theorem 4.1 it is enough to show that  $L_t$  is Fredholm for all  $t \in (a, b)$ . To this aim, by Lemma 4.3 we just need to prove that  $P_t T P_t$  is Fredholm on  $H_t$ . Clearly, the kernel and cokernel of the projection  $P_t$  are  $H_t^\perp$ , which is of finite dimension  $k < +\infty$ , and hence  $P_t$  is a Fredholm operator. Whence,  $P_t T P_t$  is Fredholm, because the composition of Fredholm operators is still Fredholm (cf. [9, Theorem 3.2]).  $\square$

*Proof of Theorem 3.3.* Also in the setting of Theorem 3.3 we can apply Theorem 4.1 once we prove that  $L_t$  is Fredholm for all  $t \in [a, b]$ . Unluckily, since  $k = n = +\infty$ , none of the arguments used in the proof of Theorem 3.2 can be applied. Instead, by the (new) hypothesis on  $T$ , taking any  $t \in [a, b]$  the assertion follows from the remark here below:

$$L_t = P_t T P_t + P_t^\perp = P_t(I_H + K)P_t + P_t^\perp = P_t + P_t K P_t + P_t^\perp = I_H + P_t K P_t,$$

which is a compact perturbation of  $I_H$  since the set of compact operators is an ideal in  $\mathcal{L}(H)$ . Consequently,  $L_t$  is Fredholm by a classical result of Riesz and Schauder (i.e., compact perturbations of the identity are Fredholm operators, cf. [9, Corollary XII.2.5]).  $\square$



For the proof of Corollary 3.4, we need a special case of the Implicit Function Theorem in Banach spaces that we recall here for completeness (for more details, cf. [2, Subsection 2.2]).

**Theorem 4.4.** *Let  $X, Y$  be Banach spaces and  $F : [a, b] \times X \rightarrow Y$  a continuous map. Assume that the equation*

$$F(t, x) = 0 \tag{4.3}$$

*has a solution  $(t_0, x_0) \in (a, b) \times X$ , and that the derivative  $d_x F_t$  of  $F_t := F(t, \cdot)$  with respect to  $x \in X$  exists and depends continuously on  $(t, x) \in [a, b] \times X$ . If  $d_{x_0} F_{t_0} \in GL(X, Y)$ , then there exists a neighbourhood  $U \times V \subset [a, b] \times X$  of  $(t_0, x_0)$  and a continuous map  $f : U \rightarrow V$  such that*

- $f(t_0) = x_0$ ,
- $F(t, f(t)) = 0$  for all  $t \in U$ ,
- every solution of (4.3) in  $U \times V$  is of the form  $(t, f(t))$ .

*Proof of Corollary 3.4.* Both in the hypotheses of Theorem 3.2 and in those ones of Theorem 3.3, we know that a bifurcation point  $t^* \in (a, b)$  exists. Arguing by contradiction, assume that  $\text{im}(T|_{H_{t^*}}) \cap H_{t^*}^\perp = \{0\}$ . Then, by (4.2) we have  $\ker L_{t^*} = \{0\}$ , so  $L_{t^*}$  is invertible since it is Fredholm of index 0.

On the other hand, we can define the map

$$F : (t, u) \in [a, b] \times H \mapsto F(t, u) = d_u f_t \in \mathcal{L}(H, \mathbb{R}).$$

By assumption,  $F(t, 0) = 0$  for all  $t \in [a, b]$ . Denoting  $F_t(u) := F(t, u)$ , since  $d_0 F_{t^*}(u)(v) = \langle L_{t^*} u, v \rangle$ ,  $u, v \in H$ , and  $L_{t^*}$  is invertible, we obtain that  $d_0 F_{t^*} : H \rightarrow \mathcal{L}(H, \mathbb{R})$  is invertible, too. Consequently, by Theorem 4.4 all solutions of the equation  $F(t, u) = 0$  in a neighbourhood of  $(t^*, 0) \in [a, b] \times H$  are of the form  $(t, 0)$ ; hence,  $t^*$  is not a bifurcation point of critical points of  $f_t$ , in contradiction with the fact that it is a bifurcation point for  $\mathcal{J}$  along  $\{H_t\}_{t \in [a, b]}$ .  $\square$

## 5 An example

Throughout this section, for notational simplicity we put  $I := [0, 1]$  and we denote by  $H_0^1(I, \mathbb{R}^n)$  the Hilbert space of all absolutely continuous functions  $u : I \rightarrow \mathbb{R}^n$  such that the derivative  $u'$  is square integrable, and by  $(H^{-1}(I, \mathbb{R}^n), \|\cdot\|_{H^{-1}})$  its dual space.

Our aim is investigating the existence of nontrivial solutions for the semilinear system of ordinary differential equations

$$\begin{cases} -(A(x)u'(x))' + g(x, u(x)) = 0, & x \in I, \\ u(0) = u(1) = 0, \end{cases} \tag{5.1}$$

where  $A : I \rightarrow GLS(n, \mathbb{R})$  is a smooth family of invertible symmetric matrices, and  $g : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $g = g(x, t)$ , is a  $C^1$  function such that  $g(x, 0) = 0$  for all  $x \in I$ .

Let us consider the functional  $\mathcal{J} : H_0^1(I, \mathbb{R}^n) \rightarrow \mathbb{R}$  such that

$$\mathcal{J}(u) = \frac{1}{2} \int_0^1 \langle A(x)u'(x), u'(x) \rangle dx + \int_0^1 G(x, u(x)) dx,$$

where  $G : I \times \mathbb{R}^n \rightarrow \mathbb{R}$  is any function such that  $\nabla_\xi G(x, \xi) = g(x, \xi)$ ,  $(x, \xi) \in I \times \mathbb{R}^n$ . It is well known (see, e.g., [19, Proposition B.34]) that  $\mathcal{J}$  is of class  $C^2$  in  $H_0^1(I, \mathbb{R}^n)$  and

$$d_u \mathcal{J}(v) = \int_0^1 \langle A(x)u'(x), v'(x) \rangle dx + \int_0^1 \langle g(x, u(x)), v(x) \rangle dx \tag{5.2}$$

for any  $u, v \in H_0^1(I, \mathbb{R}^n)$ . Whence, the critical points of  $\mathcal{J}$  are precisely the weak solutions of problem (5.1).

In particular,  $0 \in H_0^1(I, \mathbb{R}^n)$  is a critical point, and one can show that the corresponding Hessian is given by

$$d_0^2 \mathcal{J}[u, v] = \int_0^1 \langle A(x)u'(x), v'(x) \rangle dx + \int_0^1 \langle S(x)u(x), v(x) \rangle dx, \quad \text{for all } u, v \in H_0^1(I, \mathbb{R}^n),$$

where  $S(x) = \frac{\partial g}{\partial t}(x, 0)$  is a family of symmetric matrices.

Let us recall that, for every  $t \in I$ , we can consider the evaluation map

$$ev_t : u \in H_0^1(I, \mathbb{R}^n) \mapsto ev_t(u) = u(t) \in \mathbb{R}^n.$$

It is a bounded linear operator which is surjective if  $t \in (0, 1)$ . Moreover,  $ev_t$  depends continuously on  $t$  in  $(0, 1)$ . Indeed, for every  $t_0 \in (0, 1)$  and  $u \in H_0^1(I, \mathbb{R}^n)$ , we have

$$u(t) = u(t_0) + \int_{t_0}^t u'(s) ds, \quad t \in I,$$

which implies that

$$\|ev_t - ev_{t_0}\|_{H^{-1}} \leq \sqrt{|t - t_0|}.$$

Consequently, for every  $0 < a < b < 1$  by Lemma 2.2 we get a continuous family of subspaces  $\{H_t\}_{t \in [a, b]}$  by

$$H_t = \ker ev_t = \{u \in H_0^1(I, \mathbb{R}^n) : u(t) = 0\}.$$

**Definition 5.1.** We say that  $t^* \in (a, b)$  is a *bifurcation point* for (5.1) if there exist two sequences  $(t_n)_n \subset [a, b]$  and  $(u_n)_n \subset H_0^1(I, \mathbb{R}^n)$  such that

- (i)  $t_n \rightarrow t^*$  in  $[a, b]$  and  $u_n \rightarrow 0$  in  $H_0^1(I, \mathbb{R}^n)$  as  $n \rightarrow +\infty$ ;
- (ii)  $u_n \not\equiv 0$  for each  $n \in \mathbb{N}$ ;
- (iii) taking any  $n \in \mathbb{N}$ , the restriction  $u_{0,n} := u_n|_{[0, t_n]}$  satisfies

$$-(A(x)u'_{0,n}(x))' + g(x, u_{0,n}(x)) = 0, \quad x \in [0, t_n];$$

- (iv) taking any  $n \in \mathbb{N}$ , the restriction  $u_{1,n} := u_n|_{[t_n, 1]}$  satisfies

$$-(A(x)u'_{1,n}(x))' + g(x, u_{1,n}(x)) = 0, \quad x \in [t_n, 1],$$

- (v)  $u_{0,n}(t_n) = u_{1,n}(t_n) = 0$  for each  $n \in \mathbb{N}$ .

Let us remark that, according to Definition 5.1, the two restrictions  $u_{0,n}$  and  $u_{1,n}$  define a global solution of (5.1) if and only if  $u'_{0,n}(t_n) = u'_{1,n}(t_n)$ .

**Lemma 5.2.**  $t^* \in (a, b)$  is a bifurcation point of (5.1) if and only if it is a bifurcation point of  $\mathcal{J}$  along  $\{H_t\}_{t \in [a, b]}$ .

*Proof.* If  $t^* \in (a, b)$  is a bifurcation point of (5.1), then  $(t_n)_n \subset [a, b]$  and  $(u_n)_n \subset H_0^1(I, \mathbb{R}^n)$  exist which satisfy the properties (i)–(v) in Definition 5.1. Hence, for all  $v \in H_{t_n}$  we have that

$$\int_0^{t_n} \langle A(x)u'_{0,n}(x), v'(x) \rangle dx + \int_0^{t_n} \langle g(x, u_{0,n}(x)), v(x) \rangle dx = 0$$

and

$$\int_{t_n}^1 \langle A(x)u'_{1,n}(x), v'(x) \rangle dx + \int_{t_n}^1 \langle g(x, u_{1,n}(x)), v(x) \rangle dx = 0.$$

Then, by (5.2) it follows that  $u_n \in H_0^1(I, \mathbb{R}^n)$  is a non-trivial critical point of  $\mathcal{J}|_{H_{t_n}}$ . Since  $u_n \rightarrow 0$ ,  $t^*$  is a bifurcation point of  $\mathcal{J}$  along  $\{H_t\}_{t \in [a, b]}$  (see Definition 3.1).

Conversely, let  $(t_n)_n \subset [a, b]$  and  $(u_n)_n \subset H_0^1(I, \mathbb{R}^n) \setminus \{0\}$  be such that  $u_n \in H_{t_n}$  is a critical point of  $\mathcal{J}|_{H_{t_n}}$ , with  $t_n \rightarrow t^*$  and  $u_n \rightarrow 0$  in  $H_0^1(I, \mathbb{R}^n)$ . Setting  $u_{0,n}$  and  $u_{1,n}$  as in (iii) and (iv) of Definition 5.1, we see that

$$\int_0^{t_n} \langle A(x)u'_{0,n}(x), v'(x) \rangle dx + \int_0^{t_n} \langle g(x, u_{0,n}(x)), v(x) \rangle dx = 0 \quad \text{for all } v \in H_0^1([0, t_n], \mathbb{R}^n)$$

and

$$\int_{t_n}^1 \langle A(x)u'_{1,n}(x), v'(x) \rangle dx + \int_{t_n}^1 \langle g(x, u_{1,n}(x)), v(x) \rangle dx = 0 \quad \text{for all } v \in H_0^1([t_n, 1], \mathbb{R}^n).$$

Whence,  $u_n$  satisfies both (iii) and (iv) of Definition 5.1, while (v) is an immediate consequence of the definition of  $H_{t_n}$ . Thus,  $t^*$  is a bifurcation point of (5.1).  $\square$

As from Lemma 5.2 the existence of bifurcation points of (5.1) can be reduced to the study of bifurcation points of the functional  $\mathcal{J}$  on  $\{H_t\}_{t \in [a, b]}$ , we assume that the bilinear form  $d_0^2 \mathcal{J}$  is non-degenerate both on  $H_a$  and  $H_b$ . This implies that the path  $\{H_t\}_{t \in [a, b]}$  is admissible. Moreover, it is easy to check that the corresponding operator  $T : H_0^1(I, \mathbb{R}^n) \rightarrow H_0^1(I, \mathbb{R}^n)$  is given by

$$\begin{aligned} Tu(x) &= \int_0^x A(t)u'(t) dt - x \int_0^1 A(t)u'(t) dt \\ &\quad - \int_0^x \int_0^t S(\tau)u(\tau) d\tau dt + x \int_0^1 \int_0^t S(\tau)u(\tau) d\tau dt. \end{aligned}$$

Finally, we introduce the orthogonal projection  $P_t$  onto  $H_t$ . To this aim, we choose a smooth function  $\chi : [0, 1] \rightarrow [0, 1]$  such that  $\chi(0) = \chi(1) = 0$  and  $\chi|_{[a, b]} \equiv 1$ . Then, taking

$$Q_t u(x) := u(x) - \chi(x)u(t),$$

we have that  $Q_t$  defines a bounded projection in  $H_0^1(I, \mathbb{R}^n)$  onto  $H_t$ , so we obtain a family of projections  $P_t$  onto  $H_t$  by

$$P_t = Q_t Q_t^* (Q_t Q_t^* + (I_{H_0^1(I, \mathbb{R}^n)} - Q_t^*)(I_{H_0^1(I, \mathbb{R}^n)} - Q_t))^{-1}$$

which are orthogonal (cf. [5, Lemma 12.8 a])).

Consequently, writing down the path  $L_t = P_t T P_t + P_t^\perp$ , we have everything at hand in order to claim the existence of a bifurcation for (5.1) by Theorem 3.2 if we can show that  $\text{sf}(L, [a, b]) \neq 0$ . However, in order to avoid too many technical computations, we restrict to the special case of positive definite matrices  $A(x)$ . Then,  $L_t \in \Phi_S^+(H_0^1(I, \mathbb{R}^n))$  and according to (iv) in Section 2 we can deduce the existence of a bifurcation point if  $\mu_{\text{Morse}}(L_a) \neq \mu_{\text{Morse}}(L_b)$ . Since these Morse indices can be computed directly from the quadratic form associated to the Hessian of  $\mathcal{J}$ , we summarise our discussion for positive definite  $A$  in the following proposition.

**Proposition 5.3.** *Assume that the matrices  $A(x)$ ,  $x \in I$ , are positive definite, and that  $0 < a < b < 1$  are such that the restrictions of the Hessian  $d_0^2 \mathcal{J}$  to both  $H_a$  and  $H_b$  are non-degenerate. If*

$\sup \dim\{V \subset H_a : d_0^2 \mathcal{J}[u, u] < 0, u \in V \setminus \{0\}\} \neq \sup \dim\{V \subset H_b : d_0^2 \mathcal{J}[u, u] < 0, u \in V \setminus \{0\}\},$   
*then there is a bifurcation point for (5.1).*

Finally, let us mention that a related bifurcation problem is studied in [17, 18, 22]. There, the authors consider the Dirichlet problem for elliptic partial differential equations

$$\begin{cases} -\Delta u + g(x, u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

on a smooth bounded domain  $\Omega \subset \mathbb{R}^N$ , which is assumed to be star-shaped with respect to  $0 \in \mathbb{R}^N$ . Setting

$$\Omega_t := \{tx \in \mathbb{R}^N : x \in \Omega\} \quad \text{for all } t \in [a, 1],$$

for some  $0 < a < 1$ , they study the bifurcation along the subspaces  $\{H_0^1(\Omega_t, \mathbb{R})\}_{t \in [a, 1]}$  of  $H_0^1(\Omega, \mathbb{R})$ . However, our Theorems 3.2 and 3.3 cannot be applied to such a problem, since the spaces  $H_0^1(\Omega_t, \mathbb{R})$  do not vary continuously with respect to the metric on  $\mathcal{G}(H_0^1(\Omega, \mathbb{R}))$ . Indeed, if  $0 < s < t < 1$ , then there is a function  $u \in H_0^1(\Omega_t, \mathbb{R})$  such that  $\|u\| = 1$  and with support in  $\Omega_t \setminus \Omega_s$  (here,  $\|\cdot\|$  is the standard norm in  $H_0^1(\Omega, \mathbb{R})$ ). Consequently,  $\langle u, v \rangle_{H_0^1(\Omega, \mathbb{R})} = 0$  for all  $v \in H_0^1(\Omega_s, \mathbb{R})$  and so

$$\|P_{H_0^1(\Omega_t, \mathbb{R})} - P_{H_0^1(\Omega_s, \mathbb{R})}\|_{H^{-1}(\Omega, \mathbb{R})} \geq \|P_{H_0^1(\Omega_t, \mathbb{R})}u - P_{H_0^1(\Omega_s, \mathbb{R})}u\| = \|P_{H_0^1(\Omega_t, \mathbb{R})}u\| = \|u\| = 1,$$

which clearly contradicts the continuity.

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